

TIME AND FREQUENCY DOMAIN SOLUTION FOR
MULTI-DIMENSIONAL INCIDENT WAVES IN MULTILAYERED ABSORBING
MEDIA WITH AN APPLICATION TO SOURCE DETERMINATION

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ABSTRACT

Solutions of the ray propagation equation are obtained for various boundary conditions, assuming vertical incidence in plane, parallel, equal-travel-time layers. The solutions are examined in both the time and frequency domains and certain properties derived. A complete discussion of frequency-domain synthesis techniques is given in connection with a treatment of the absorption problem. FORTRAN programs are given which compute any of the solutions in either the frequency or time domains, with or without absorption. The theory and programs are applied to the problem of source depth determination, and it is shown that the method of pP spectral nulls is somewhat unreliable.

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INTRODUCTION

Considerable work on ray propagation in multilayered media has been reported in the literature (see Claerbout, 1968, for references). Most of it has been concerned with calculating near-source seismograms for vertically incident waves in lossless media. Claerbout summarized previous results for this case in the concise notation of Sherwood and Troke (1965), and included in his paper a computer program to synthesize reflection seismograms in the time domain. He also derived the solution for the transmission seismogram due to a source buried in the lower half-space. Landers and Claerbout (1969) applied the half-space transmission solution to a study of crust/upper mantle models proposed by Aki. Frasier (1970) has developed solutions for non-vertically-incident P and SV waves in non-absorptive media which are closely analogous to Claerbout's.

A totally satisfying treatment of the absorptive case has not been given, to the author's knowledge. Troke (1962) devised a time-domain solution using a non-realizable linear absorption law. For practical reasons he was forced to lump his absorption into a small number of constant-Q bands. His procedure seemed cumbersome and no attempt was made by this author to duplicate it. Instead, a frequency-domain approach has been utilized. Several authors have expressed the fear that this would lead to large aliasing errors, but such has not proved to be the case. A discussion of practical difficulties is given. Besides its simplicity, the frequency-domain calculation has the additional advantage that there is no particular problem in making one's absorption law realizable. Sherwood and Troke (1965) gave essentially a physical argument that the delayed

transmission seismogram must be minimum phase for a minimum-phase absorption law. A mathematical proof is presented here. A discussion of dispersion is also given.

Since this work was completed, a paper has appeared (Jensen and Ellis, 1970) in which the authors obtain solutions for non-vertical incidence plus absorption, using linear system theory. They calculate spectra, not seismograms, and they do not include dispersion. No programs are given. An alternative approach to the general problem would be to apply the techniques developed here to the solutions given by Frasier.

The emphasis in the present work has been placed on obtaining algorithms which are fast, accurate, and concise, and therefore useful for large-scale model studies. The first part of the paper is devoted to obtaining lossless transmission solutions for various boundary conditions of interest (assuming vertical incidence in plane, parallel, equal-travel-time layers). A computer program is included to do these calculations (exactly) in the time domain. The second part considers the practical aspects of frequency-domain absorption calculations. FORTRAN programs are included to do all the cases treated in the first part in the frequency domain, including frequency- and depth-dependent absorption. Versions both with and without dispersion are given.

One application of the theory and programs, to source depth determination, is discussed in detail. Sources of error in determinations by P-wave spectral nulls or cepstral analysis are investigated both theoretically and experimentally. It is shown, in particular, that inhomogeneities near the source can produce large shifts in the null frequencies from the values predicted using a simple echo model. Further, it appears that there is no

really practical way to circumvent this effect. Thus a limit can be placed on the reliability of this method.

Certain other applications of the programs are mentioned briefly.

THE LOSSLESS CASE

Propagation equation

For up- and downgoing displacement waves U and D , waves at the surface are related to those at depth by the "propagator matrix"

$$\begin{bmatrix} U \\ D \end{bmatrix}_{\text{surface}} = \frac{1}{w^k \prod t} \begin{bmatrix} z^k F(1/z) & z^k G(1/z) \\ G(z) & F(z) \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \quad (1)$$

Derivation of (1) is given in detail by Claerbout and will not be repeated here. (Note that for convenience we factor out a transmission factor $\prod t \equiv \prod_{i=1}^k t_i$.) F and G are polynomials in z , obtained by taking products of k "layer matrices"

$$\begin{bmatrix} U \\ D \end{bmatrix}_j = \frac{1}{wt} \begin{bmatrix} z & zr \\ r & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_{j+1} \quad (2)$$

where $z = w^2 = e^{-i\omega\tau}$ represents a unit delay operator, and τ is the two-way travel-time across each layer (the same for all layers) and equals the sampling interval. The (displacement) reflection coefficients r are defined at each interface by

$$r = (\rho_a v_a - \rho_b v_b) / (\rho_a v_a + \rho_b v_b) \quad (3)$$

for densities ρ and velocities v , a above and b below; and the (downgoing) transmission coefficients t are related to the reflection coefficients by $t = 1 + r$. Equation (1) may be conveniently viewed in either the time or frequency domains.

In this paper we consider only problems in which at least one free surface is present, and it turns out that in this case the quantity of interest in (1) is not F or G , but the combination

$$A(z) \equiv F(z) - z^k G(1/z) \quad (4)$$

(A is normalized and is related to Claerbout's M by $A(z) = \prod_i t_i M(z)$.) A recursion for A may be developed by multiplying the propagator for $k-1$ layers by an additional layer matrix

$$\begin{bmatrix} z^{k-1}F(1/z) & z^{k-1}G(1/z) \\ G(z) & F(z) \end{bmatrix} \begin{bmatrix} z & zr_k \\ r_k & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .. & r_k z^k F(1/z) + z^{k-1} G(1/z) \\ .. & r_k z G(z) + F(z) \end{bmatrix}$$

from which

$$A^{(k)}(z) = [r_k z G(z) + F(z)] - [r_k z^k F(1/z) + z^{k-1} G(1/z)]$$

by definition (4), or

$$A^{(k)}(z) = A^{(k-1)}(z) - r_k z^k A^{(k-1)}(1/z) \quad (5)$$

Claerbout gives essentially this expression. In the time-domain this is equivalent to the recursion

$$A_1^{(k)} = A_1^{(k-1)} = \dots = 1$$

$$A_j^{(k)} = A_j^{(k-1)} - r_k A_{k-j+2}^{(k-1)} ; \quad j = 2, \dots, k$$

$$A_{k+1}^{(k)} = -r_k$$

(obtained by putting $A^{(k)}(z) = A_1 + A_2 z + A_3 z^2 + \dots + A_{k+1} z^k$ into (5) and identifying coefficients of powers of z).

Half-space

For the case of an observer located at a free surface and an impulsive source buried in a half-space below the layers, the boundary conditions are a perfect reflection of the observed seismogram $X(z)$ at the surface, a 1 coming up from below, and some function $P(z)$ returned into the half-space. The propagation equation (1) is thus

$$\begin{bmatrix} X \\ X \end{bmatrix} = \frac{1}{w^k \pi t} \begin{bmatrix} z^k F(1/z) & z^k G(1/z) \\ G(z) & F(z) \end{bmatrix} \begin{bmatrix} 1 \\ P \end{bmatrix} \quad (6)$$

Inverting and solving for X yields

$$X(z) = w^k \prod_{i=1}^k (1-r_i)/A(z) \quad (7)$$

(The determinant in (6) is $z^k F(1/z)F(z) - z^k G(1/z)G(z) = z^k \prod_i (1-r_i^2)$, from (2).) This is Claerbout's solution. The seismogram due to an arbitrary source function $S(z)$ may be obtained by convolving the impulse response with S .

To actually compute (7) we first compute A by means of the recursion (5) and then invert. Note that in the programs we do our convolutions (multiplications) and deconvolutions (divisions) in the time domain for accuracy. In the time domain, convolution and deconvolution are N^2 processes (require a time proportional to N^2 for N elements), but since the layer recursion (5) is also N^2 , little is lost by doing this. Note also that the factor w^k in (7) corresponds to the initial delay of the first arrival, but this is ignored in the programs and output commences with the first point.

Source on far surface

If both observer and source are located at free surfaces, boundary conditions are as shown in Figure 1.

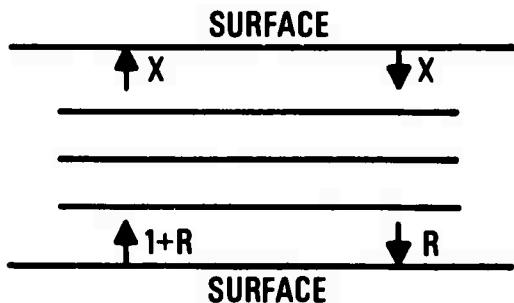


Figure 1. Source on far surface

R here is some unknown function. The propagation equation for this case is

$$\begin{bmatrix} X \\ x \end{bmatrix} = \frac{1}{w^k t} \begin{bmatrix} z^k F(1/z) & z^k G(1/z) \\ G(z) & F(z) \end{bmatrix} \begin{bmatrix} 1 + R \\ R \end{bmatrix}$$

Solving for X,

$$X(z) [A(z) - z^k A(1/z)] = w^k \Pi_i (1-r_i) \quad (8)$$

Inspection of (5) and (7) shows that the same answer could be obtained from the half-space solution by adding an extra layer with $r = 1$ (an obvious result), provided that the layer is not included in the transmission factor $\Pi_i (1-r_i)$. Writing $A^*(z) \equiv A(z) - z^k A(1/z)$, (a definition that will be used throughout this paper), (8) becomes

$$X(z) = w^k \Pi (1-r_i) / A^*(z) \quad (8')$$

The inverse wavelet in this case is antisymmetric about its midpoint. This implies, among other things, that half the information about the layers is lost, that is, one cannot go from the seismogram back to the reflection coefficients of the layers in this case. Another way to see this is as follows. Inverting z in (5), multiplying by $r_k z^k$ and adding to A gives

$$(1-r_k^2) A^{(k-1)}(z) = A^{(k)}(z) + r_k z^k A^{(k)}(1/z) \quad (9)$$

which is the inverse recursion that allows us to extract the reflection coefficients from the inverse wavelet. In (9) we remove layers one at a time, where at each step $r_k = -A_{k+1}^{(k)}$ is the last boundary. Evidently (9) blows up when any $r_k = \pm 1$.

This result has an interesting physical interpretation. Consider the propagator of an inverted layer set. If the normal case is written

$$\begin{bmatrix} U' \\ D' \end{bmatrix} = Q \begin{bmatrix} U \\ D \end{bmatrix}$$

we write the inverted case

$$\begin{bmatrix} D \\ U \end{bmatrix} = \bar{Q} \begin{bmatrix} D' \\ U' \end{bmatrix}$$

by inspection of Figures 2 and 3.

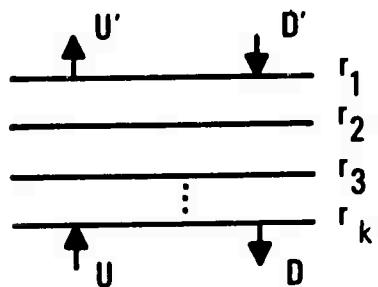


Figure 2. Normal case

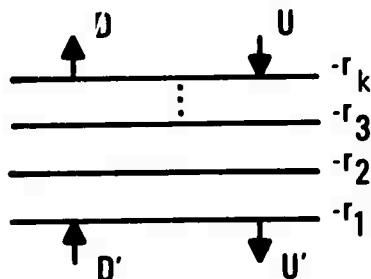


Figure 3. Layers inverted

Comparison with (1) gives

$$\bar{Q}(z) = \frac{1}{w^k \pi(1-r_i)} \begin{bmatrix} z^k F(1/z) & -G(z) \\ -z^k G(1/z) & F(z) \end{bmatrix} \quad (10)$$

This can also be obtained from (1) by a time reversal, $z \rightarrow 1/z$, plus a matrix inversion, i.e. $\bar{Q}(z) = Q^{-1}(1/z)$. Writing $\bar{A}(z) \equiv F(z) + G(z)$, by analogy with (4), we have

$$\begin{aligned} \bar{A}^*(z) &= \bar{A}(z) - z^k \bar{A}(1/z) \\ &= [F(z) + G(z)] - z^k [F(1/z) + G(1/z)] \\ &= A(z) - z^k A(1/z) \end{aligned}$$

or

$$\bar{A}^*(z) = A^*(z) \quad . \quad (11)$$

Thus the antisymmetric combination is invariant under an inversion of the layers. Comparing this with (8'), we see that interchanging source and receiver gives the same seismogram, except for a scale factor (reciprocity), i.e. one

cannot tell, from the seismogram, which side of the earth is which.

In principle, a reflection coefficient slightly less than unity removes the difficulty associated with (9), but in practice the requirement is that $1-r^2$ must not be so small as to make the recursion unstable. In any case, the actual reflection coefficient of the distant free surface is just the ratio of the last to first points of the inverse wavelet. The presence of absorption will change these results drastically. In fact the inverse problem is then not do-able even in the half-space case (at least not using the techniques discussed here), essentially because one is required to obtain $2k$ items of information from only k items of data.

Contained source

Of some interest is the case of a source buried in a layer stack terminated at both ends by free surfaces. This case may be easily treated by coupling half-space solutions back to back (Figure 4).

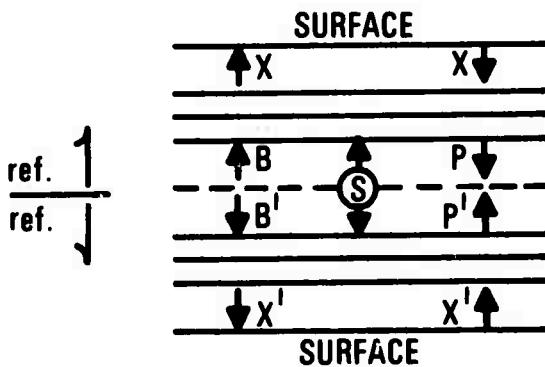


Figure 4. Buried source

We assume the source to be located in a trivial ($r = 0$) interface common to the two layer stacks. The transmitted wave due to a source $B(z)$ is, by (7),

$$X(z) = w^k \Pi_i (1-r_i) B(z)/A(z) \quad (7')$$

The wave returned from the half-space may be obtained by solving (6) for P

$$P(z) = w^k A(1/z) X(z) / \Pi_i (1-r_i) \quad (12a)$$

$$= z^k B(z) A(1/z) / A(z) \quad (12b)$$

Primes will be used to denote quantities associated with the distant stack. From Figure 4, the coupling relations are

$$B(z) = S(z) - P'(z) \quad (13)$$

$$B'(z) = S(z) - P(z) \quad (13')$$

for a spatially symmetric source $S(z)$. The minus signs in (13) are necessary to take account of the change in reference direction of the (displacement) waves between the two layer sets. Substituting (12a) into (13'), (13') into (12b) (primed), and the result into (13) gives

$$B(z) = S(z) - z^{k'} [S(z) - \frac{w^k}{\prod(1-r_i)} A(1/z)X(z)] \frac{A'(1/z)}{A'(z)}$$

Using (7') and collecting terms

$$\begin{aligned} & [A(z)A'(z) - z^{k+k'} A(1/z)A'(1/z)] X(z) \\ &= w^k \prod_{i=1}^{k-1} (1-r_i) [A'(z) - z^{k'} A'(1/z)] S(z) \end{aligned} \quad (14)$$

Note that the filter on the LHS of (14) is antisymmetric and depends on all the layers, whereas the filter on the RHS, which is also antisymmetric, depends only on the layers lying between the source and the far surface.

A comparison with the far-surface solution (8) is tempting, but in that case one reference direction was used for all layers, whereas in this case we have decomposed the stack into two opposing substacks. To make the comparison, we should write the propagator for the whole stack in terms of the two substacks. This is just the product of the propagator for the near stack and the propagator for the far stack inverted, i.e. $Q_T = Q \circ \bar{Q}'$. From (1) and (14) this is

$$\begin{aligned} Q_T &= \frac{1}{w^k \prod t} \begin{bmatrix} z^k F(1/z) & z^k G(1/z) \\ G(z) & F(z) \end{bmatrix} \\ &\cdot \frac{1}{w^{k'} \prod (1-r'_i)} \begin{bmatrix} z^{k'} F'(1/z) & -G'(z) \\ -z^{k'} G'(1/z) & F'(z) \end{bmatrix} \\ &= \frac{1}{w^{k+k'} \prod (1+r_i) \prod (1-r'_i)} \begin{bmatrix} \dots & -G'(z)z^k F(1/z) + F'(z)z^k G(1/z) \\ \dots & -G'(z)G(z) + F'(z)F(z) \end{bmatrix} \end{aligned}$$

from which

$$A_T(z) = F'(z) A(z) + G'(z) z^k A(1/z)$$

by (4), and

$$\begin{aligned} A_T^* &\equiv A_T(z) - z^{k_T} A_T(1/z) \\ &\equiv A(z) A'(z) - z^{k+k'} A(1/z) A'(1/z) = (AA')^* \end{aligned} \quad (15)$$

Putting this result in (14) gives

$$A_T^*(z) X(z) = w^k \prod_{i=1}^{k-1} (1-r_i) A'^*(z) S(z) \quad (14')$$

Comparison of (14') with (8') shows that the only difference is in the RHS of (14'), that is, one obtains the same seismogram from a source S_s located at the far surface as from a buried source S_b provided

$$S_s(z) = \frac{A'^*(z)}{w^k \prod_{i=1}^{k-1} (1+r_i')} S_b(z) \quad (15)$$

(The change in the sign of the reflection coefficients arises

because we have considered the primed stack to be ordered from the far surface down. Actually, theorem (11) says that we may calculate A'^* from the reversed stack as well, in order to be consistent with the calculation of A_T^* . This is done in the programs). Thus (15) represents a "source-burying operator" which describes the effect of burying the source in terms of the layers between the far surface and the source.

Remembering that A'^* is antisymmetric about its midpoint, which represents the depth of the source in travel-time (sampling-rate) units, we might hope to use this fact in source depth determination. If one could isolate A'^* by some means, then the least-squares point of antisymmetry would give the depth. For example, if a seismogram X_s were available from an identical source lying along the same path at the surface, then comparison with the buried-source seismogram X_b would give

$$X_b/X_s \sim A'^*$$

ignoring scale and delay factors. This method suffers from certain obvious disadvantages. A more direct approach would be to look for zeros of X_b . This has been done in the past (cepstrum analysis) and suffers principally from the effect of the source window $S(z)$. A study of the limitations of the method is presented below under Applications. An alternative derivation of the buried-source operator given in the next section shows that the result is general.

Source-burying operator:

An alternative derivation of the source-burying operator can be given by comparing (distant) half-space solutions for a far-surface source and requiring that the incoming and outgoing waves be identical for the two cases (Figures 5 and 6).

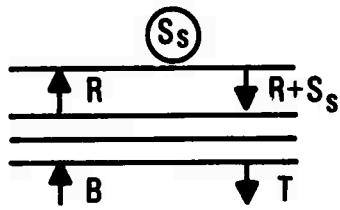


Figure 5. Surface source

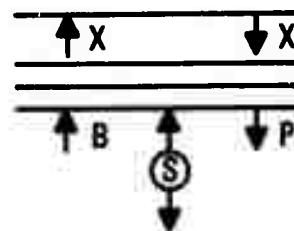


Figure 6. Buried source

For a surface source, the propagation equation (1) is

$$\begin{bmatrix} R \\ R + S_s \end{bmatrix} = \begin{bmatrix} 1 & z^k F(1/z) & z^k G(1/z) \\ w^k \pi t & G(z) & F(z) \end{bmatrix} \begin{bmatrix} B \\ T \end{bmatrix}$$

Solving for T gives

$$T = \frac{w^k \pi(1+r_i)}{A(z)} S_s + \frac{z^k A(1/z)}{A(z)} B \quad (16)$$

The solution for the buried-source case is given by (12b) with source B-S (the minus being necessary to keep the reference direction consistent with the previous case), i.e.

$$P = [z^k A(1/z) / A(z)] (B-S)$$

Then

$$P + S = \frac{A(z) - z^k A(1/z)}{A(z)} S + \frac{z^k A(1/z)}{A(z)} B \quad (17)$$

Identifying T in (16) with P + S in (17) gives

$$S_s = \frac{A(z) - z^k A(1/z)}{w^k \pi(1+r_i)} S \quad (15)$$

as before.

In the buried-source solutions we have so far considered only the case of a source located within a homogeneous layer. If the source is located in a reflecting interface, we can modify the above derivation by explicitly considering the reflections at the additional interface. If this boundary has reflection

coefficient c , instead of (15) we obtain

$$S_s = \frac{A(z) - (1+2c) z^k A(1/z)}{w^k \Pi(1+r_i)} s \quad (18)$$

This resembles the previous result with, however, a surface reflection coefficient $r = 1+2c$. (This can easily be proved by inverting the stacks in Figures 5 and 6 and going through the calculation assuming a surface reflection coefficient r other than unity).

We can also modify the previous derivation to include the case of a spatially asymmetric source. If in Figure 6 we replace the upcoming (toward the surface) part of the source function by $S'(z)$, then it is easy to show that (15) becomes

$$S_s = \frac{A(z)S(z) - z^k A(1/z)S'(z)}{w^k \Pi(1+r_i)} \quad (19)$$

If $S(z)$ and $S'(z)$ have identical time behavior but differ by a scale factor (for example, an earthquake with an asymmetrical radiation pattern) then (19) will have the form of (18). The programs have not been written to handle these cases, but it would be easy to modify them to do so, simply by redefining the value of the terminal reflection coefficient between calls to the inner computation routine (RCTOA).

Buried-receiver operator

The results we have obtained have assumed the receiver to

be located at the surface. A "buried-receiver operator", analogous to the buried-source operator, can be derived which relates the seismogram received by a buried receiver to that received by a surface receiver lying on the same path. This operator depends only on the layers lying between the buried receiver and the surface and could be used to relate the signals received by sensors in a vertical array.

Consider a sensor located in a trivial boundary which terminates the upper layer set, as shown in Figure 7.

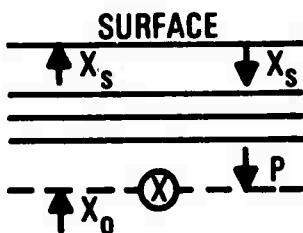


Figure 7. Buried receiver

For a wave $x_0(z)$ coming from below and a wave $P(z)$ returning from above, the observed seismogram $X(z)$ will be the sum (all are displacement waves), i.e.

$$X(z) = x_0(z) + P(z)$$

By (12b)

$$P = \frac{z^k A(1/z)}{A(z)} x_0$$

and from (7')

$$x_s = \frac{w^k \Pi(1-r_i)}{A(z)} x_0$$

Combining we get

$$x = \frac{A(z) + z^k A(1/z)}{w^k \Pi(1-r_i)} x_s \quad (20)$$

This differs from (15) in that the buried-receiver operator (20) is symmetric. The discussion associated with (9) still applies, that is, we should not expect to be able to go from the ghosting filter for our array back to the reflection coefficients of the layers. Inspection of the derivation leading to (11), however, shows that a similar result does not hold for the symmetric case, i.e. the symmetric combination is not invariant under an inversion of the layers.

As with the buried-source operator, we can remove the restriction that the sensor be located within a homogeneous layer and allow it to be located at a non-trivial interface. This case is somewhat less interesting than the corresponding

extension for the source operator, but it is easy to do. The result

$$x = \frac{A(z) + (1-c) z^k A(1/z)}{w^k \pi (1-r_i)(1-c)} x_s \quad (21)$$

if the interface containing the receiver has reflection coefficient c.

ABSORPTION AND THE FREQUENCY-DOMAIN CALCULATION

So far we have viewed our solutions in the time domain, but we could equally well view them in the frequency domain. The z-transform formulation gives us the vantage point of Janus in being able to see both worlds at once. This is the power of the equal-travel-time assumption. While it is not strictly necessary to retain this assumption in the frequency-domain calculation, it is convenient for our purposes to do so. Considering simplicity, speed, and accuracy, the time-domain solution is undoubtedly the correct approach in the absence of absorption; but since we now wish to treat the general case of frequency- and depth-dependent absorption, a frequency-domain approach will henceforth be necessary. (See Trorey, 1962, for an example of what is encountered in attempting a time-domain solution for this case). The transition to the frequency domain is easily made by noting the definition $z = e^{-i\omega\tau}$. Then instead of representing polynomials in z , our expressions can be considered to be complex functions of frequency. Since we wish a discrete, sampled time function with sampling interval τ and will be using a Discrete Fourier Transform (DFT) to obtain it from the frequency function, we will take as our phase interval $\Delta\omega\tau = 2\pi/M$, where M is the number of time or frequency points desired (see Rader and Gold, 1969, for a discussion of Discrete Fourier Transform theory, and Sherwood and Trorey, 1965, for a discussion of the z-transform).

Propagator with absorption

Following Sherwood and Trorey, we can obtain the layer matrix from (2) by noting that z represents the two-way delay in crossing the layer. With absorption present, the wave will be further modified. If we represent our (one-way) absorption law by $f(z)$, then replacing z by $zf^2(z)$ and w by $wf(z)$ gives the layer matrix with absorption

$$\begin{bmatrix} U \\ D \end{bmatrix}_j = \frac{1}{wtf(z)} \begin{bmatrix} zf^2(z) & zf^2(z)r \\ r & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_{j+1} \quad (22)$$

We then write the propagator (1) in the form

$$\begin{bmatrix} U \\ D \end{bmatrix}_{\text{surface}} = \frac{1}{w^k \pi_i t_i f_i(z)} \begin{bmatrix} F'(z) & G'(z) \\ G(z) & F(z) \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \quad (23)$$

where F' and G' are related to F and G . (We take Claerbout's form for our layer matrix, rather than Sherwood and Trorey's. In ours, the absorptive band f_k lies above the boundary r_k). Again, we will be interested only in the combinations

$$\begin{aligned} A(z) &\equiv F(z) - G'(z) \\ A'(z) &\equiv F'(z) - G(z) \end{aligned} \tag{24}$$

defined by analogy with (4). We develop recursions for A and A' in the same manner as in the lossless case. Multiplying the propagator by an additional layer matrix,

$$\begin{aligned} & \begin{bmatrix} F'(z) & G'(z) \\ G(z) & F(z) \end{bmatrix} \cdot \begin{bmatrix} zf_k^2(z) & zf_k^2(z)r_k \\ r_k & 1 \end{bmatrix} \\ = & \begin{bmatrix} zf_k^2 F' + r_k G' & zf_k^2 r_k F' + G' \\ zf_k^2 G + r_k F & zf_k^2 r_k G + F \end{bmatrix} \end{aligned}$$

from which we obtain

$$A_k(z) = [F(z) - G'(z)] + zf_k^2(z)r_k [G(z) - F'(z)]$$

$$A'_k(z) = r_k [G'(z) - F(z)] + zf_k^2(z) [F'(z) - G(z)]$$

or

$$\begin{aligned} A_k(z) &= A_{k-1}(z) - zf_k^2(z)r_k A'_{k-1}(z) \\ A'_k(z) &= -r_k A_{k-1}(z) + zf_k^2(z) A'_{k-1}(z) \end{aligned} \tag{25}$$

corresponding to (5). (To avoid conflict, we now use subscripts to indicate the iterative step.) In the absence of absorption, $f_k(z) = 1$, and it is easy to show from (25) that in this case $A'_k(z) = z^k A_k(1/z)$, which gives back (5). At each frequency, (25) represents a set of coupled recursions over layers in the complex quantities A , A' , z , and f . Initially, $A_1(z) = A'_1(z) = 1$.

Since we require a real time function $A(t)$, $A(\omega)$ must be conjugate-symmetric about zero frequency. This implies that we need only calculate $A(\omega)$ for positive frequencies (up to the folding phase, π). This also implies that $f(z)$ must be conjugate-symmetric (the "crossing symmetry" relations of Futterman, 1962) in order that (25) will lead to a conjugate symmetric function.

We will take as our absorption law

$$f_k(\omega) = e^{-|\omega\tau|/4Q_k} + i\phi_k(\omega) \quad (26)$$

where $\phi(\omega)$ represents the dispersive contribution to the phase, and Q conforms to the usual definition of the quality factor. This "linear absorption law" has been experimentally verified in rocks (see Troprey, 1962, for references) and appears to be the best choice for a calculation of this type. Neglecting dispersion (discussed below), $f = \exp(-|\omega\tau|/4Q)$ and we can use efficient recursions over frequency to calculate the exponential and the sines and cosines in (25).

We see here the difficulty with the time-domain calculation of (25). For each layer added, we must do two complete

convolutions. Thus this procedure is N^3 . For small absorption, our filter will have short effective length, so we might still get by. The alternative is to use Troprey's approach. In either case, one is also faced with the necessity of obtaining a time representation of (26) for each layer.

Absorptive solutions

Knowing how to calculate A , we can immediately transcribe all our solutions to include absorption by the technique used to obtain (22). We list the results here for reference.

$$X = w^k \prod_i^k (1-r_i) f_i(z) / A(z); \text{ half-space} \quad (27)$$

$$X = w^{k_T} \prod_i^{k_T} (1-r_i) f_i(z) / A_T^*(z); \text{ far-surface} \quad (28)$$

$$X = w^k \prod_i^k (1-r_i) f_i(z) B^*(z) / A_T^*(z); \text{ buried-source} \quad (29)$$

$$X = B^*(z) / w^k \prod_i^k (1-r_i) f_i(z); \quad \text{b. source operator} \quad (30)$$

$$X = A^+(z) / w^k \prod_i^k (1-r_i) f_i(z); \quad \text{b. receiver operator} \quad (31)$$

To avoid conflict we have used B to refer to the distant layer stack, which we have also inverted (stacked away from observer) so as to be consistent. A careful calculation of the

buried-source operator for this case shows (29) and (30) to be the correct forms. Here $A^* \equiv A(z) - z f_k(z) A'(z)$, which is calculated by addition of the far surface $r_k = 1$ (which must not be included in the transmission factors in 28 and 30). Evidently A^* is no longer antisymmetric. Similarly, $A^+ \equiv A(z) + z f_k(z) A'(z)$.

Proof of minimum-phase

Claerbout gives an inductive proof based on (5) that $A(z)$ must be minimum phase in the lossless case. (See Sherwood and Troke for a discussion of the terms "minimum phase", "realizable", and "positive-real" and their application to seismic problems). By a slight modification of his argument, it is possible to show directly from (25) that $A(z)$ must in fact be minimum phase for any realizable absorption law $f(z)$. We rewrite (25) in the form

$$\frac{A_k(z)}{A_{k-1}(z)} = 1 - z r_k f_k^2(z) \frac{A'_{k-1}(z)}{A_{k-1}(z)} \quad (32)$$

$$\frac{A'_k(z)}{A'_{k-1}(z)} = z f_k^2(z) - r_k \frac{A_{k-1}(z)}{A'_{k-1}(z)} \quad (33)$$

Assuming $f(z)$ has no poles inside the unit circle (i.e. is realizable), then we need only to show that $|A'/A| \leq 1$ on the unit circle to complete Claerbout's proof. (It is evident from

the structure of (25) that $A'(z)$ must be realizable when f is.)

$$\underline{\text{Lemma}}: |A'(z)/A(z)| \leq 1 \text{ when } |z| = 1 \quad (34)$$

Proof: We prove (34) by induction. Writing $\alpha_k(z) \equiv A'_k(z)/A_k(z)$, then dividing (33) by (32) gives

$$\alpha_k = \alpha_{k-1} \left[\frac{zf_k^2 - r_k/\alpha_{k-1}}{1 - zr_k f_k^2 \alpha_{k-1}} \right] = - \frac{r_k - zf_k^2 \alpha_{k-1}}{1 - zr_k f_k^2 \alpha_{k-1}}$$

We must show that $|\alpha_k| \leq 1$ when $|\alpha_{k-1}| \leq 1$, i.e., we require

$$|r_k - zf_k^2 \alpha_{k-1}| \leq |1 - zr_k f_k^2 \alpha_{k-1}| \quad (35)$$

given that $|z| = 1$, $|f_k^2| \leq 1$, and $|r_k| \leq 1$. We establish (35) by contradiction, i.e., assume

$$|r_k - zf_k^2 \alpha_{k-1}|^2 > |1 - zr_k f_k^2 \alpha_{k-1}|^2$$

or

$$|z f_k^2 \alpha_{k-1}|^2 + r_k^2 - 2 \operatorname{Re}(r z f^2 \alpha)$$

$$> r_k^2 |z f_k^2 \alpha_{k-1}|^2 + 1 - 2 \operatorname{Re}(r z f^2 \alpha)$$

Since $|z| = 1$, we have

$$|f_k^2 \alpha_{k-1}|^2 + r_k^2 > r_k^2 |f_k^2 \alpha_{k-1}|^2 + 1$$

If $|f_k^2 \alpha_{k-1}| = 1$, we have the desired contradiction; if not, we can only have $|f_k^2 \alpha_{k-1}| < 1$, by the assumptions. Collecting terms we get

$$r_k^2 [1 - |f_k^2 \alpha_{k-1}|^2] > 1 - |f_k^2 \alpha_{k-1}|^2$$

Since the common factor must be positive, we can divide and obtain $r_k^2 > 1$, a contradiction. Since $\alpha_1 = 1$, the lemma is provided.

Having proved the crucial addition, we restate Claerbout's proof. Assume A_{k-1} is minimum phase. Assuming f is realizable, (32) must be realizable. Further, the second term must have magnitude less than or equal to unity on the unit circle, by the Lemma. Hence its real part must be less than or equal to unity and therefore (32) must have positive real part on the unit circle. Since all A_k represent real time functions, (32) must be real when z is real. Therefore (32) is "positive-real"

and hence minimum phase. Thus A_k is minimum phase whenever A_{k-1} is. Since $A_1 = 1$, all A_k must be minimum phase.

The minimum-phase condition on $A(z)$ is necessary in order that we may obtain realizable seismograms. Inspection of (27) shows that if our absorption law $f(z)$ is also minimum phase, then we actually obtain minimum-phase seismograms, provided we remove the initial delay. This is true even in the case of two free surfaces ($r_k = 1$) (note, however, that this may introduce poles or zeros on the unit circle).

Dispersion

The key requirement in the above discussion was the physical realizability of $f(z)$. The zero-phase absorption law $f = \exp(-|\omega\tau|/4Q)$ is evidently not realizable, since real symmetric frequency functions possess real symmetric time transforms. Hence $f(t)$ possesses non-causal precursor before time $t = 0$. Futterman (1962) has derived a relation between the amplitude and phase parts of (26) based on causality (Kramers-Kronig dispersion relations). He shows that the real part of the index of refraction can be related to the imaginary part by the Hilbert transform

$$\text{Re } \Delta n(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \Delta n(\Omega)}{\Omega - \omega} d\Omega \quad (36)$$

and he obtains a simple form for the phase in the case of a linear absorption law (26). He uses the infinite Hilbert transform, but since we are performing a discrete, frequency-limited synthesis, we might expect the finite Hilbert transform

$$\phi(\omega) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f_a(\Omega)| \cot \frac{\omega-\Omega}{2} d\Omega \quad (37)$$

to work better. Such is the case. In (37) f_a represents the absorptive part of (26) and ϕ is the dispersive phase. (Note that in this section we use $\underline{\omega}$ to mean $\underline{\omega}\tau$. Obviously the arguments in (37) must be dimensionless.) It can be shown (Rader and Gold, 1969) that functions for which the phase is the Hilbert transform of the log magnitude are actually minimum phase. Thus we make our absorption law minimum phase.

Putting (26) into (37) gives

$$\phi(\omega) = \frac{1}{8\pi Q} \int_{-\pi}^{\pi} |\Omega| \cot \frac{\omega-\Omega}{2} d\Omega \quad (38)$$

for our phase. Reducing (38) to the range of positive frequencies and integrating by parts gives a term which vanishes plus a term involving $\int \log \sin u du$, for which no explicit form exists. It can, however, be expressed in terms of "Lobatchevsky's function", for which a series expansion is given by Gradshteyn and Ryzhik (1965). Carrying out the algebra gives

$$\phi(\omega) = \frac{-1}{\pi Q} \sum_{k \text{ odd}}^{\infty} \frac{\sin k\omega}{k^2} \quad (39)$$

which is exactly the result one would obtain by deriving the frequency-domain Hilbert transform (i.e. interchanging sine and cosine coefficients and changing the parity) of the

function $|\Omega|/4Q$. Thus we appear to be stuck with (39). Actually, it turns out that the most successful procedure is just to do the frequency-domain Hilbert transform using the FFT (Fast Fourier Transform). Synthesizing ϕ from (39) using the inverse FFT was slightly less successful, supporting our original supposition that when doing discrete synthesis it is best to remain entirely within the discrete realm (observe that (39) represents an infinite sum and the coefficients $1/k^2$ were obtained by doing an integral). Futterman's phase did quite a bit worse. The criteria in all these cases was the observed realizability of $f(t)$.

Several properties of the discrete dispersive phase are immediately evident from (39), namely $\phi(0) = \phi(\pi) = 0$. Furthermore, ϕ has an extremum at $\pi/2$ about which it is symmetric:

$$\phi\left(\frac{\pi}{2}\right) = -\frac{1}{\pi Q} \sum_{k=0}^{\infty} (-1)^k / (2k+1)^2 = -G/\pi Q \quad (40)$$

where $G \approx .91596559\dots$ is Catalan's constant and $G/\pi \approx .2915609\dots$ Using these properties, an empirical approximation to (39) was obtained,

$$\phi(\omega) \approx -G \log [1 + \omega(\pi-\omega)] / \pi Q \log (1 + \pi^2/4) \quad (41)$$

While this form was not used in the programs, it might be useful in some applications. It fits fairly well, but actually it is not much more trouble just to compute the Hilbert transform.

Having a procedure to compute the phase, we have all the elements needed to complete our frequency synthesis. Using a dispersive absorption law in (25) requires the explicit computation of all the sines and cosines and consequently more computing time. For comparison, routines were written with and without dispersion in $A(z)$. In both cases, dispersion was included in the straight-through transmission filter (the numerator in 27-29). This is just given by (26) with a dissipation factor $\sum_i 1/Q_i$, which can be large for many layers, even in the case of high Q . Comparison of the semi-dispersive and complete dispersive calculations is given in Figure 8. Running time for the dispersive calculation was only ~1.5 times as long as for the semi-dispersive case, but the semi-dispersive routines have also been included here because of their simplicity.

Inversion methods

We have delayed until now a discussion of the chief difficulty with the frequency-domain calculation: aliasing, as it is called by most authors. A function that is synthesized in the frequency domain and is not time-limited will have a time transform in which later times are folded over into earlier times, since the computed spectrum is the spectrum of the entire record and since the DFT treats time functions as though they were periodic (Rader and Gold). Several authors have mentioned the effect in connection with this problem, but evidently none has diagnosed the real difficulty. The key, it turns out, lies in the method of performing the deconvolutions in (27-31). From time-domain theory, we know that the inverse wavelet $A(t)$ has a finite length equal to the number of layers (in the lossless case), whereas the seismogram $X(t)$ has in principle infinite length (and in the case of two free surfaces, i.e. (28) and (29), does not decay at all). This indicates that

it would be better to transform the numerator and denominator, $\prod_i f_i(z)$ and $A(z)$, into the time domain and do a time-domain deconvolution than to divide in the frequency domain and then transform. Using the first procedure we should get zero aliasing error in the lossless case, provided we take at least as many frequencies, M , as layers, K . In fact, comparing the output from the time and frequency routines, LAYERS and FLAYERS, with infinite Q and random reflection coefficients, gave agreement to 8-11 significant digits. Tests performed using frequency-domain deconvolution, however, gave results whose accuracy depended critically on M . The error in this is "wraparound error", due to the fact that frequency-domain deconvolution is circular, whereas time-domain deconvolution is not. Thus the real villain is wraparound, not aliasing.

In the absorptive case, $A(t)$ is not strictly finite in length, but develops a rapidly decaying tail. Thus some aliasing error will develop in this case, but in general $A(t)$ is still much better-behaved than $X(t)$, and one can make the effect negligible by choosing M large enough. Since the tail on A decreases monotonically*, one can always determine by inspection in any case whether aliasing has been significant. This is definitely not the case with X . Values of M about 20% larger than K have been used successfully for reasonable Q 's, and it would probably never be necessary to use values as high as $2K$.

*Inspection of (25) shows that the highest-order term in $A(z)$ is $z^K r_K \prod_i f_i^2(z)$, so that the tail behaves like the overall transmission filter with twice the dissipation factor. Thus for any "buffer" allowance $M-K$, the error will be a function of the total dissipation factor.

Another point that should be mentioned here concerns the way (22) was factored. If, instead, the inverse absorption were left inside the matrix, then instead of (25) we would have

$$A_k(z) = f_k^{-1}(z) A_{k-1}(z) - z f_k(z) r_k A'_{k-1}(z)$$

$$A'_k(z) = -r_k f_k^{-1}(z) A_{k-1}(z) + z f_k(z) A'_{k-1}(z)$$

and in the numerators of (27-29) we would have only a scalar transmission factor. This approach is tempting because of its simplicity, but unfortunately it is numerically very unstable for low Q's. The approach we have used, however, appears to be completely stable for the first three cases (27-29), even for very low Q's. The last two cases (30, 31) are stable for reasonable Q's, but in all likelihood the "burying operators" would be computed only for rather few layers anyway.

EXAMPLES

Examples of calculated seismograms are given in Figure 8. Nonabsorptive, absorptive, and absorptive-dispersive seismograms are compared for each of four different realizations of random layer structures. Each structure consists of 100 layers terminated at each end by free surfaces. Reflection coefficients were drawn from a uniform distribution on (-.3, .3) and Q's were drawn from a "log-uniform" distribution, lying between 32 and 320, centered on 100. The source (a spike) was buried 10 layers down from the far surface. This model is admittedly not very realistic and is intended primarily to illustrate the effects of absorption. Two effects are to be noted: a general smoothing of the seismogram, due to the low-pass transmission filter in the numerator, and a progressive simplification down the record, due to the inclusion of absorption in $A(z)$. Inclusion of dispersion in A had rather little effect, although this naturally depends on the dissipation (39).

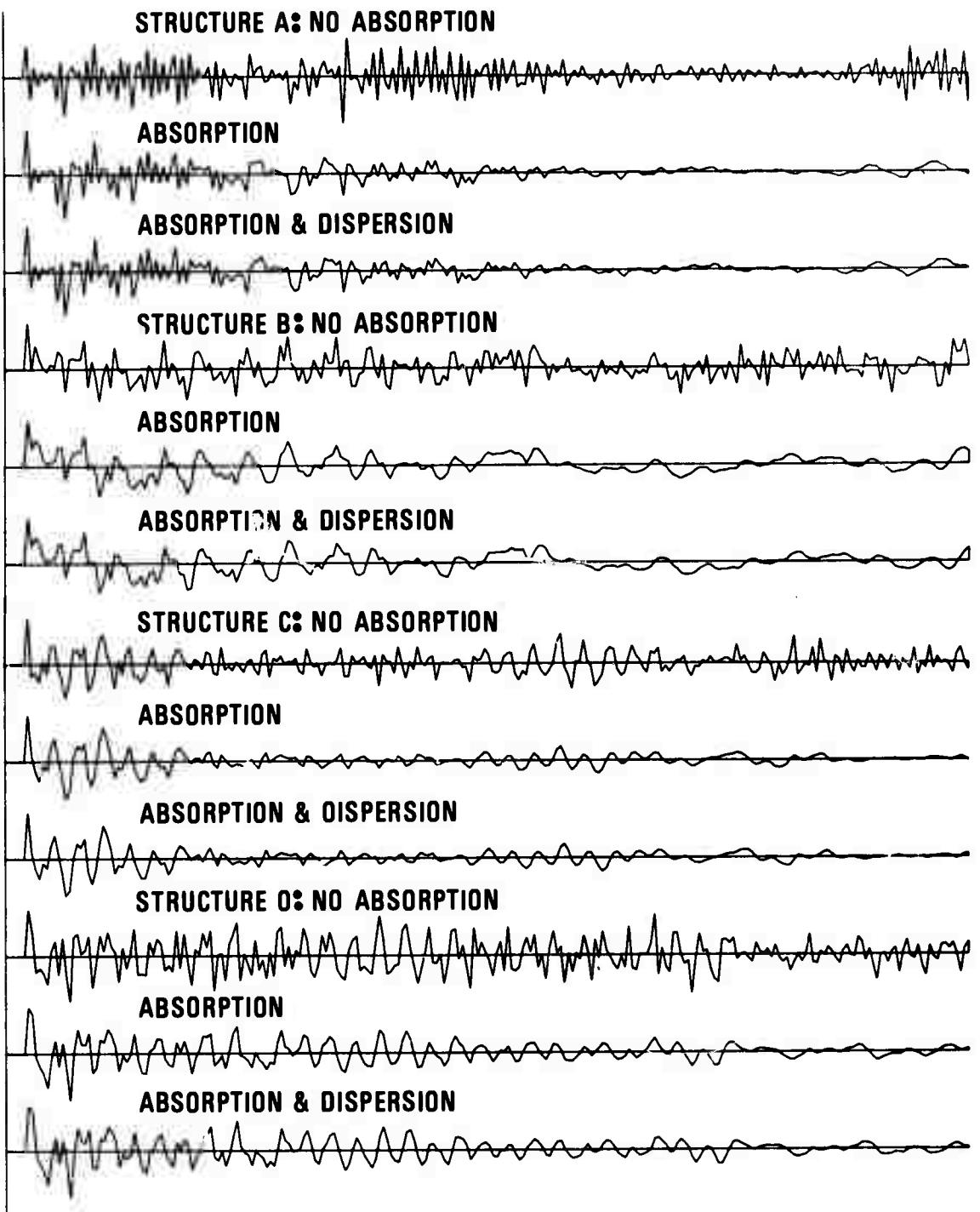


Figure 8. Computed seismograms.

36a

37

APPLICATIONS

Of several applications of the routines which have already been made, one will be described here: testing the reliability of pP depth-determination procedures. Another application, generating data to test a homomorphic filter program designed to reduce convolutional noise by beaming in the pseudo-time domain (complex cepstrum), is being made by P. R. Lintz. Quasi-realistic model studies are also being carried on by R.L. Sax. Some work has also been done on statistical behavior of synthetic seismogram spectra, which will be reported in the future.

Source depth determination

The numerator of the buried-source solution (14) contains the source depth information. One might hope to retrieve this information by an analysis of the spectral zeroes of the seismogram, since the denominator, which is the total-path filter, contributes only poles. As previously mentioned, this has been done (Cohen, 1970). The usual procedure is to smooth out the poles and enhance the visibility of the zeroes by averaging spectra from several stations for the same event.

Actually, the receiver-end effect can in practice be factored out of (14). Assuming only weak reflectors deep in the mantle (disregarding core phases), (5) says that $A(z)$ will be fairly short. The derivation leading to (15) applies to any decomposition of the path, so we may split the path in the middle. Then writing A for the layers above the source, B for all those on the source end, and C for the receiver-end layers, we can write (14)

$$x(z) \sim A^*(z) / [B(z) C(z) - z^{k_T} B(1/z) C(1/z)]$$

If both B and C are effectively short compared with the path length k_T , and if we confine our attention to the first part of the seismogram, then

$$x \sim A^*(z) / B(z) C(z)$$

Averaging over stations gives

$$x \sim [A^*(z) / B(z)] \cdot \sum_i [1/C_i(z)] \quad (42)$$

assuming that A and B are nearly the same for each station. The C_i will usually be quite different and the average will tend to smooth out the effect of these. Thus we are usually justified in ignoring receiver-end effects provided we average over stations.

Confining our attention to the numerator A^* , in the absence of reflectors this is just $1-z^k$ for a source buried k layers deep. The power spectrum is then

$$P(\omega) = |1-e^{-ik\omega\tau}|^2 = 2(1-\cos k\omega\tau)$$

which has nulls at integer multiples of $f_0 = 1/k\tau$. Thus measuring the null periodicity gives the depth of the source in time units: $t_d = 1/2f_0$. In practice, one is usually restricted to measuring the first null frequency because of the effect of the source window. Another means of determining the periodicity is to take the spectrum of the spectrum (the cepstrum; see Cohen*, 1970).

Cohen has applied the method to real data, with mixed results. Conversion of pP delay times to depths requires the application of an "average velocity" and the resultant depth estimates will evidently be only as accurate as the velocity assumed. Errors of this nature could explain the discrepancies observed by Cohen. Another possible source of error, investigated here, is the effect of reflectors on the null frequencies. It turns out that introduction of rather few layers with reasonable reflection coefficients can produce large shifts in the nulls. In principle, the cepstrum should provide a more stable estimate of the nulls, but in practice the cepstrum can also become quite messy.

The effect of reflectors can be seen by writing the source-burying operator A^* , the numerator in (42), in the form

$$A^*(z) = [1 - z^k A(1/z)/A(z)] A(z) \quad (43)$$

*Actually, Cohen zeroes out the negative-frequency part of the spectrum before transforming. It can be shown that this "cepstrum" is just the squared envelope of the autocorrelation. This is also evident from his plots. Our "cepstrum", however, is just the square of the autocorrelation.

Since $|A(1/z)/A(z)| = 1$, $A^*(z)$ will still have zeroes on the unit circle, although they will no longer be equally spaced if $A(z) \neq 1$. Evidently $A(z)$ itself will contribute no zeroes on the unit circle, by Claerbout's proof or by consideration of (5). The number of zeroes, and hence the average spacing, will therefore remain the same. Thus the cepstrum should provide a stable estimate where inspection of null spacing fails. (Large subsurface reflectors would, however, put additional zeroes close to the unit circle, which would contribute to the cepstrum in the form of peaks at shorter delay times). This is shown in Figure 9, in which the ideal case is compared with cases having inhomogeneities above the source. (The first plot in this and succeeding figures is for a homogeneous medium between the source and surface. The other threee are realizations of random structures, all with the same source-depth delay time. The random reflection coefficients in the latter cases are drawn from a uniform distribution on $[-.3, .3]$ with zero mean. There are 15 interfaces above the source, excluding the surface, i.e. this could represent a delay time of 1.6 seconds with a sampling rate of 20 sps. These and most of the following plots are 4-decade semi-log plots.) Although the null spacings vary widely, the true delay times (marked by arrows) are given in each case by the "break" in the cepstrum. If we consider the denominator in (42) also, and allow layers below the source as well, then this criterion breaks down and we see cepstral peaks at longer delay times (Figure 10: reflection coefficients above the source are the same; in addition, there are 15 below drawn from the same population). Including a source causes further complications (Figure 11). (The source function is an empirical one of Cohen's). Convolution with a source results in a multiplication (or modulation) of the cosine ripple by the source spectrum. In the cepstrum, the result is the source

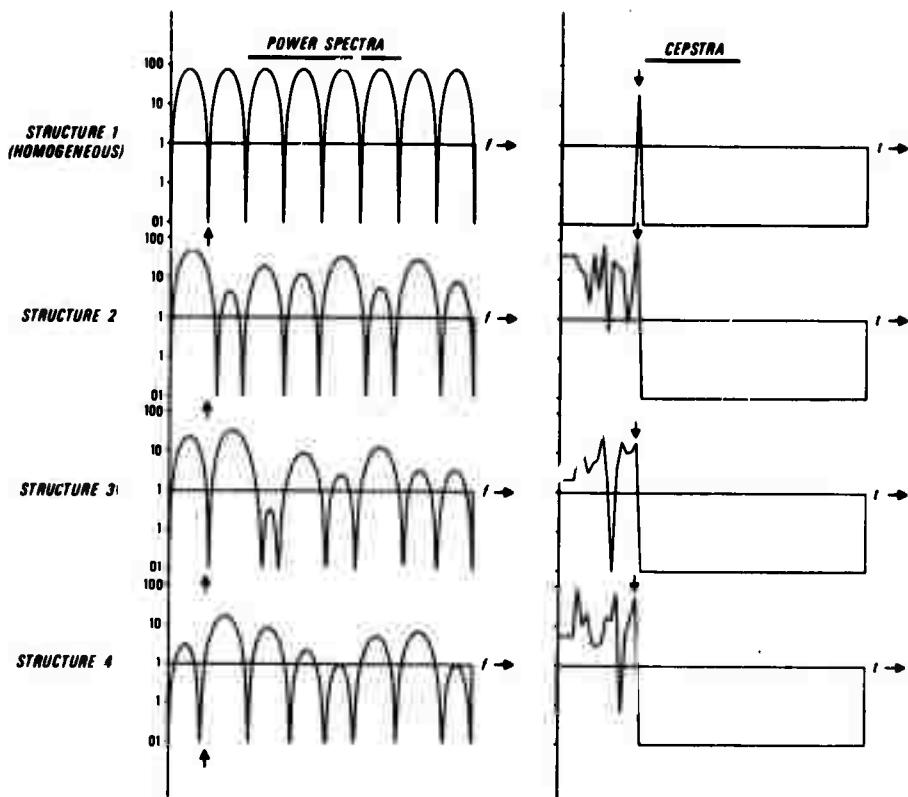


Figure 9. Layers above source.

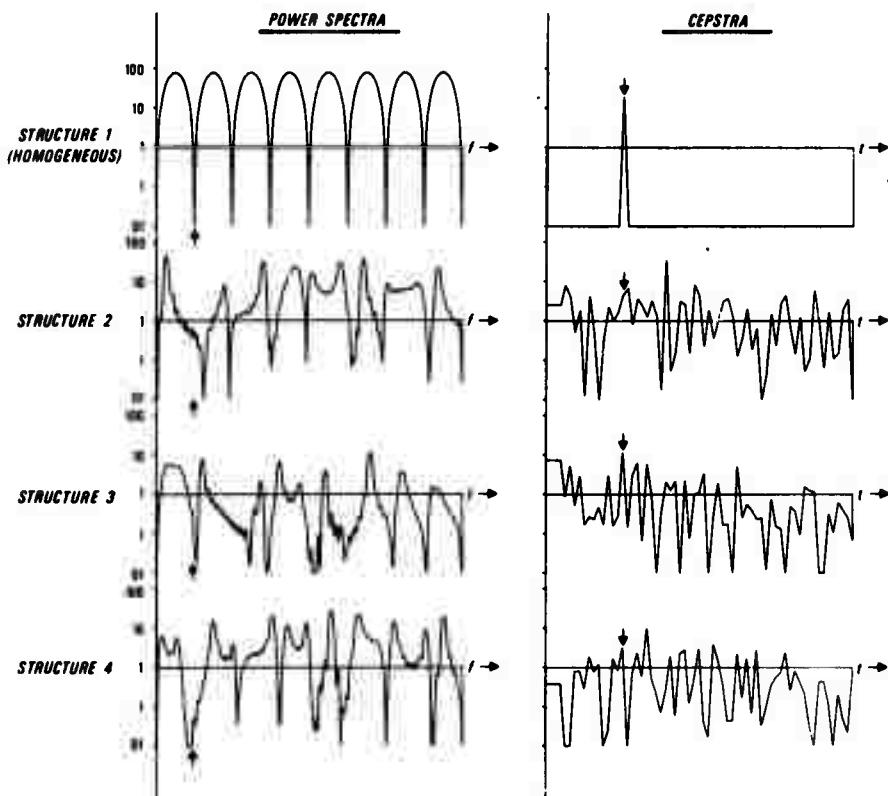


Figure 10. Layers above and below source.

40a

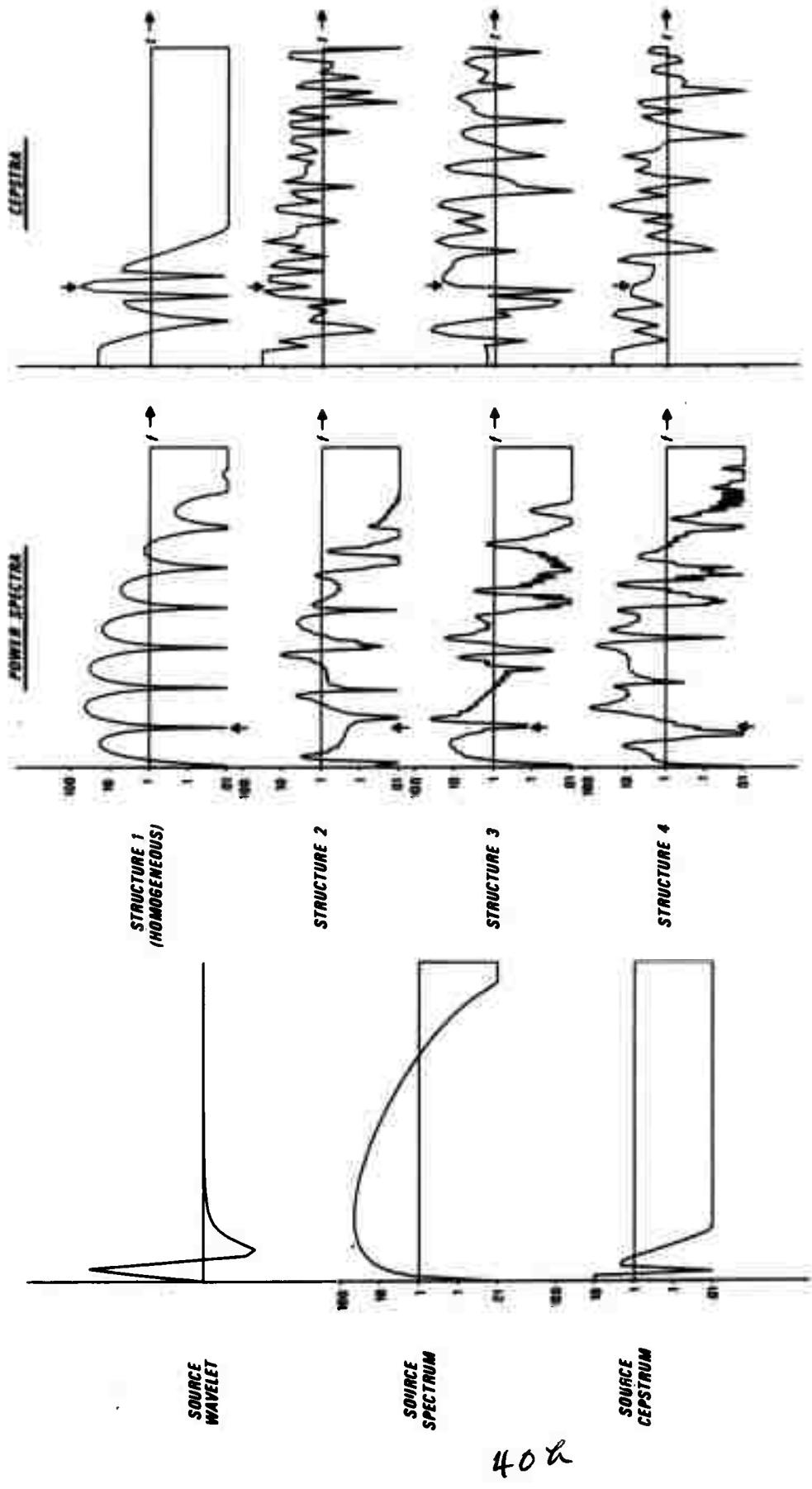


Figure 11. With source wavelet.

cepstrum plus the echo cepstrum plus the sum and difference (beats). Taking the logarithm of the spectrum before transforming makes the source and echo additive and produces the sum of the two log cepstra upon transformation. This tends to simplify the cepstrum somewhat, and appears to improve the accuracy (Figure 12).

Actually, the behavior of the log cepstra observed in Figure 12 was borne out by 47 random realizations of the same type, that is, there was almost always a sizable peak at the correct delay time, although it was not always the biggest one. (In some cases, choosing the biggest peak would lead to considerable errors.) It was expected that the presence of noise would tend to nullify any advantage possessed by the log cepstrum over the linear cepstrum. In a rough attempt to model this situation, log cepstra were computed for the same 47 realizations, with the spectra clipped at 1/30 of the maximum (~ - 15 db). The first three cases, with the standard, are shown in Figure 13. As anticipated, any superiority largely disappears. Only 14 cases of the 47 yielded the correct answer, compared with 29 correct answers in the absence of noise (these numbers are subjective). In addition, there is little tendency towards any peak at all at the correct delay time, much like the behavior of the linear cepstra (which also gave 14 correct, or nearly-correct, answers out of 47). The characters of the "noisy" log cepstra and the linear cepstra are remarkably similar.

Since our "cepstra" are really just the square of the autocorrelation, one might suppose the autocorrelation itself to be less confusing by a factor of two, since it contains phase information. (Cohen uses the product of the autocorrelation with his square-envelope autocorrelation.

This results in a peaky, smooth representation, convenient for visual analysis, but of course it is no more accurate than the autocorrelation itself.) In Figure 14 we plot the signed square of the autocorrelation for the same cases so far considered. (For convenience in plotting, the peak at zero delay is zeroed out. These are linear plots.) The correct delay time should be marked by a large negative peak. In fact, in one realization there are no large negative peaks! This is of course embarrassing, and tends to indicate the general unreliability of auto-correlation/cepstrum analysis.

Assuming then, that the ultimate arbiter must be the position of the first spectral null, since for shallow sources it is usually the only one visible, it was decided to try to determine just how variable this criterion might be. A hundred realizations of the type discussed (15 layers above the source and 15 below, reflection coefficients between -.3 and +.3, source depth fixed in time units) were drawn and the standard deviation of the observed first null position from the correct answer determined. This turned out to be about 25% of the correct answer. The maximum deviation observed is somewhat open to debate. There are several cases among the hundred where it seems unlikely that, in the presence of noise and other factors, the first null would be seen and correctly identified. Two such cases are shown in Figure 15. In all likelihood, the second nulls would be the ones picked here (remember that these are logarithmic plots). Thus it seems that this criterion could be off by a factor of at least two in some cases. Studies made with a deeper source indicate that it might be off even more, the effect being that a deep source can masquerade as a shallow one. Three such cases out of 15 are shown in Figure 16 (linear plots). The sources are 4 times

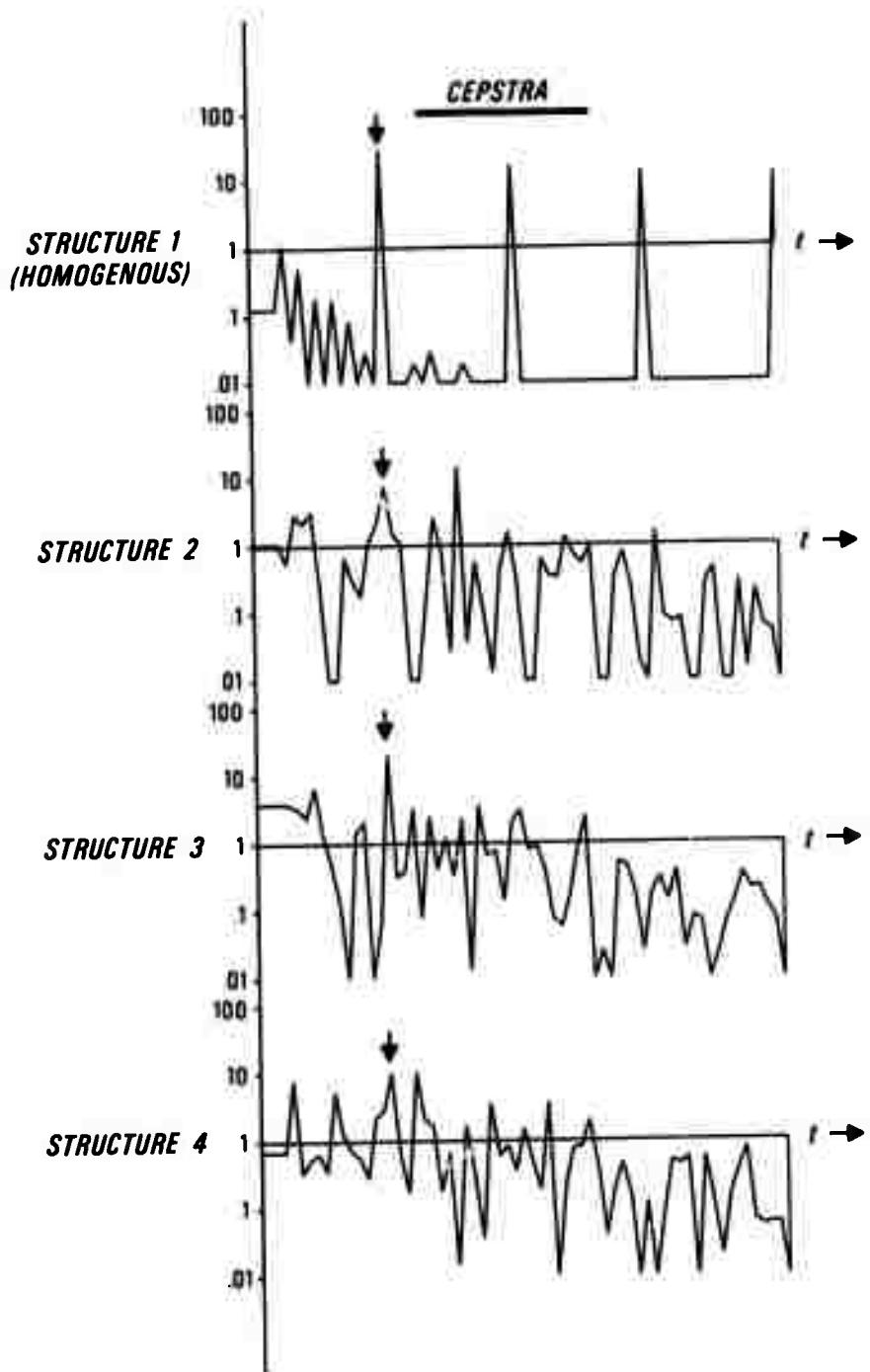


Figure 12. Log cepstra.

42a

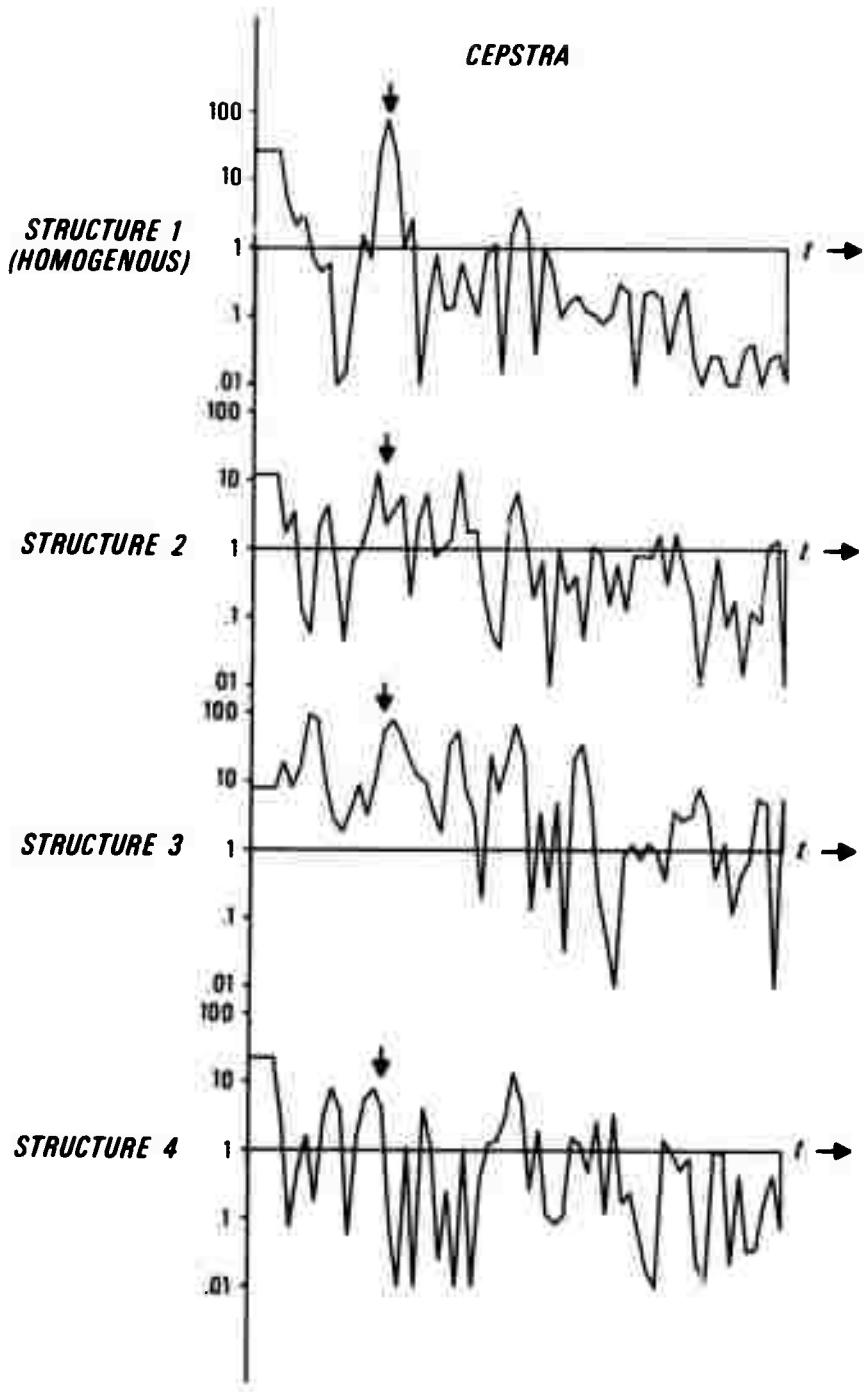


Figure 13. Log cepstra of clipped spectra.

H²K

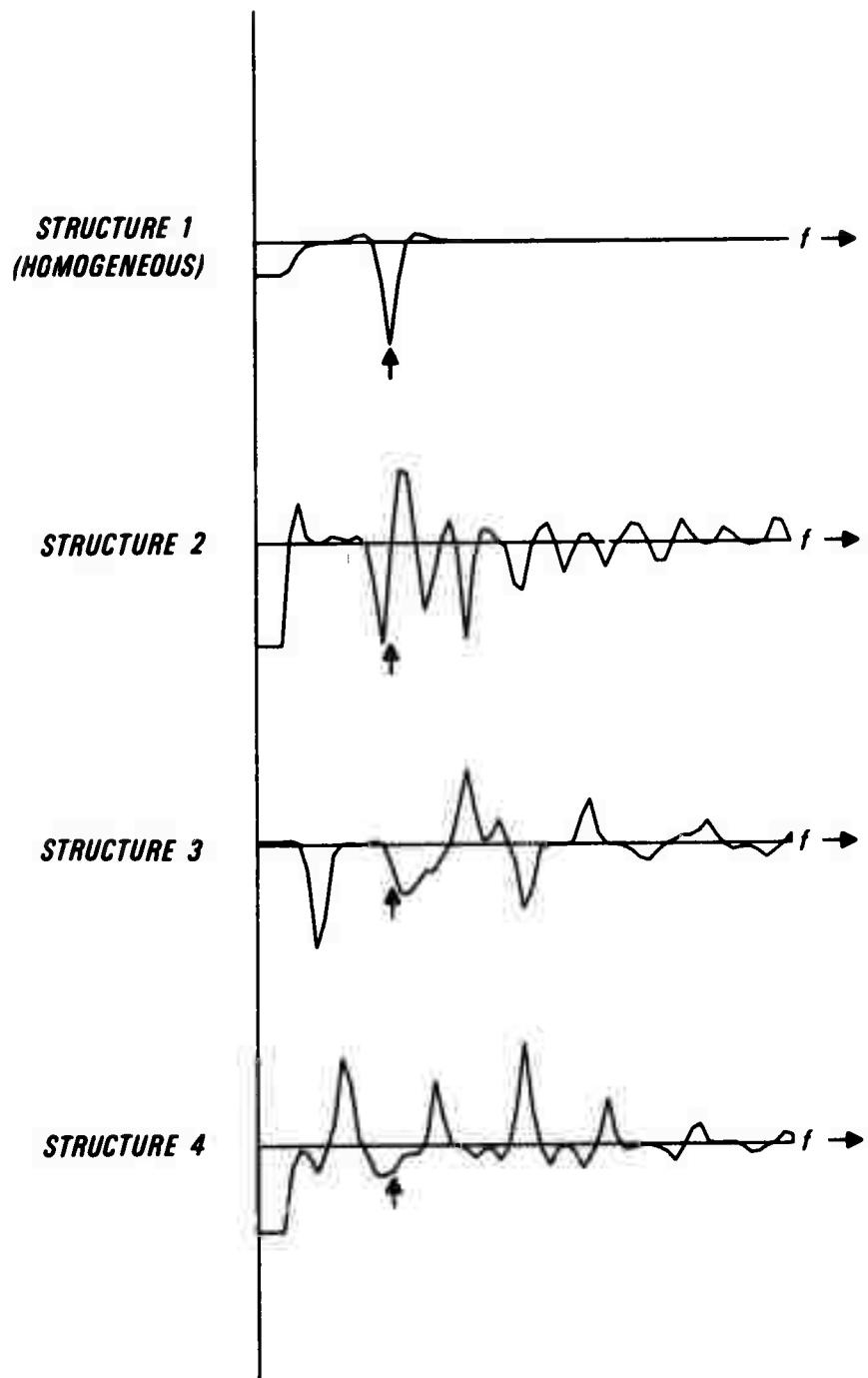


Figure 14. Signed square autocorrelation.

42c

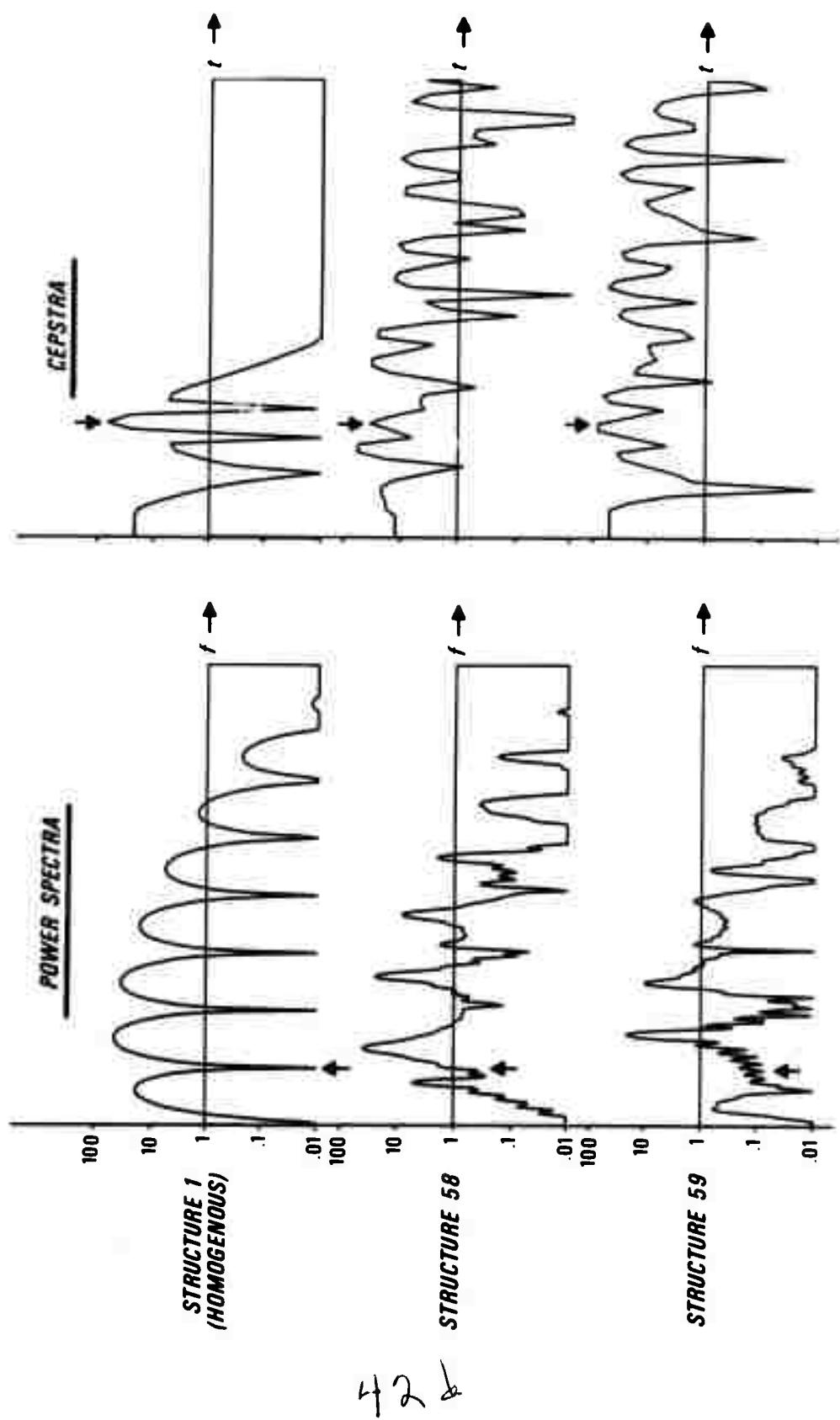
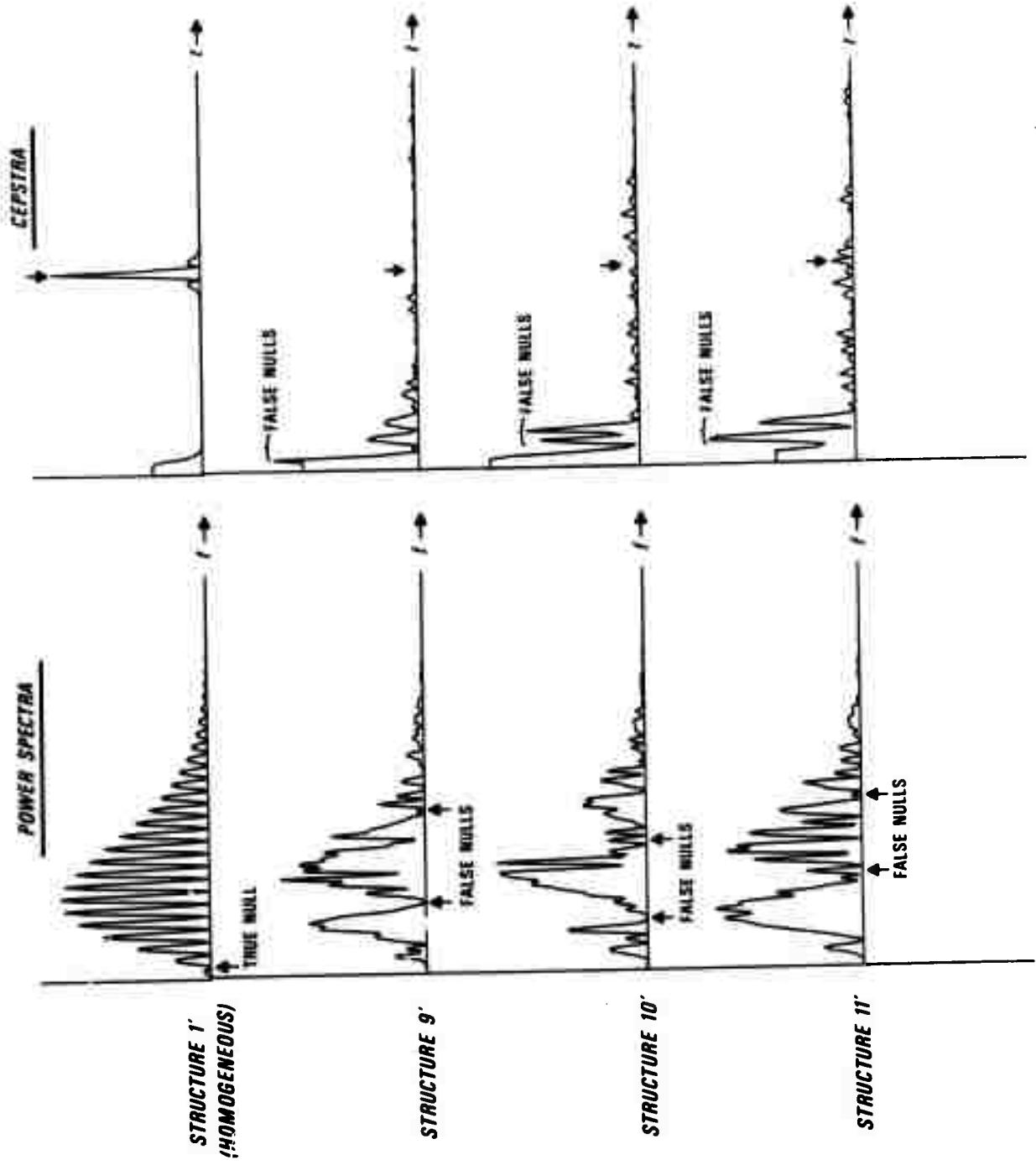


Figure 15. Two interesting cases.



42e

Figure 16. Deep source.

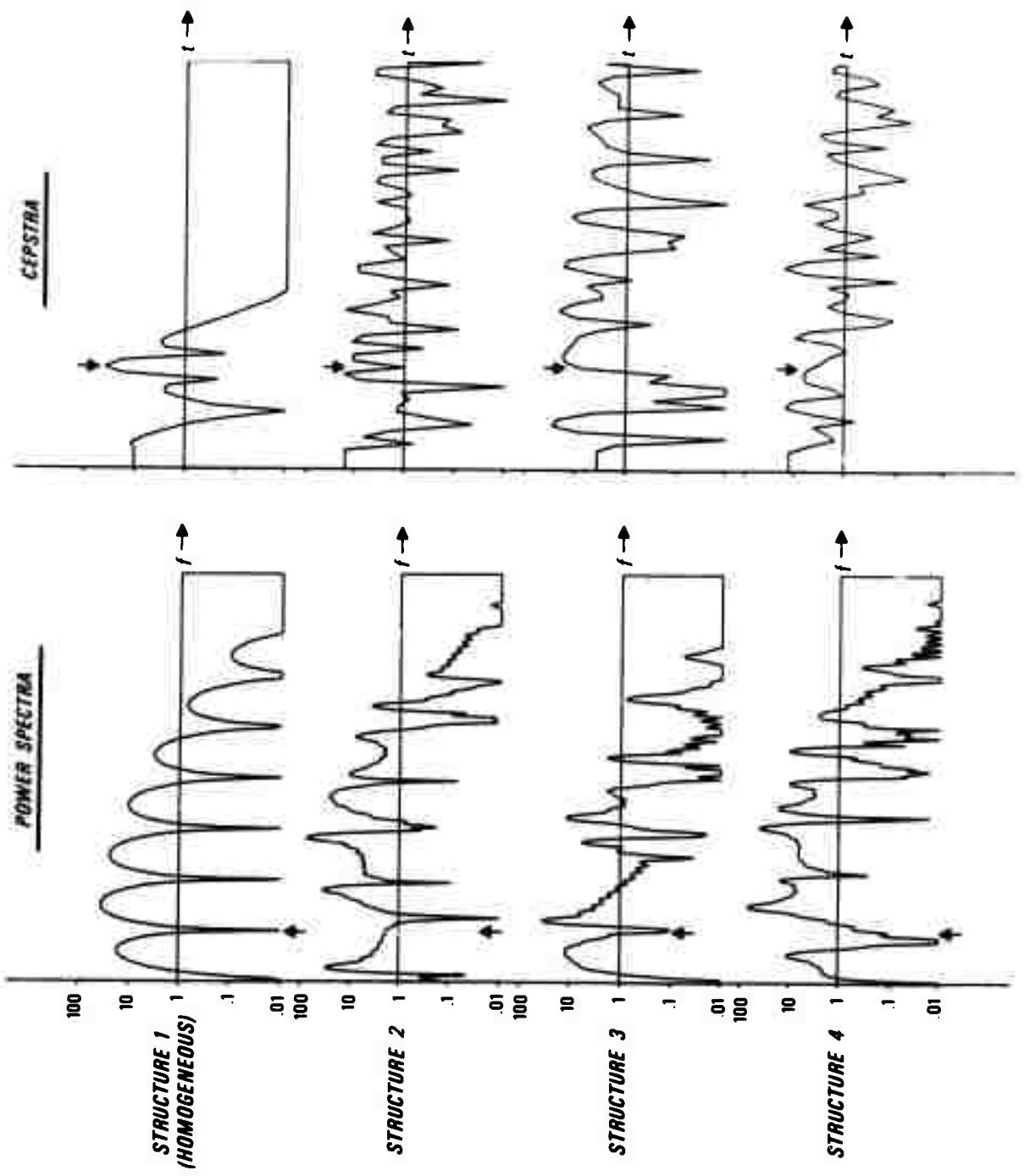


Figure 17. With absorption.

as deep as before, 63 interfaces above the source (drawn from the same population as before) and none below. A fairly broad null falsely indicating shallow depth occurs in all of these and, interestingly, the cepstrum seems to substantiate the false nulls. The question here is whether the high-frequency ripple in the spectrum would be visible enough to tip off the analyst. At least one deep earthquake has been seen with such a spectrum (Cohen, private communication).

Inclusion of absorption might be expected to weaken the nulls, but for the cases computed here, it actually had rather little effect (Figure 17). (All layers assumed to have $Q = 50$. The long transmission path through the earth was ignored, since it would not effect the low frequency nulls and since it is more or less included in Cohen's "source".) Non-vertical incidence, which is not treated in this paper, could be expected to produce further complications.

In conclusion, inhomogeneities of the type which may be found in sedimentary regions* can cause large errors in source depth delay times determined from the first spectral null. While cepstral analysis overcomes part of the difficulty, other effects can produce an extremely complicated cepstrum, difficult to interpret and inaccurate in itself. Thus this method of source depth determination would appear to be of limited utility, although the appearance of spectral nulls indicating shallow depth might be taken as corroborative evidence in the presence of other information.

*See, for example, Clark (1966). A velocity step of 2.1 to 3.9 km/sec gives a reflection coefficient of 0.3. The conclusions reached here are not intended to be rigorous, but indicative.

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I am indebted to Dr. Robert Sax for many stimulating discussions concerning the layer solutions as well as for the initial impetus. I am grateful to Dr. Theodore Cohen for information relating to his paper and for his encouragement.

APPENDIX

46a

PROGRAMS

Three packages are included: the lossless, time-domain routine LAYERS; the semi-dispersive, absorptive, frequency-domain routine FLAYERS (I); and the dispersive frequency routine FLAYERS (II). The first package comprises the routines LAYERS, RCTOA, and CONVOLV/POLYDIV. The second, FLAYERS (I), RCAFTOA, and CONVOLV/POLYDIV. The third, FLAYERS (II), RCAFTOAD, and CONVOLV/POLYDIV. The same routine CONVOLV/POLYDIV is used for all three packages. Usage of the routines is explained in the introductory "comment" statements. The two frequency packages have purposely been made interchangeable, the only difference in usage being that a storage equivalence is allowed in one that is not allowed in the other. The time routine convolves with a source function. With the frequency routines, the user may do this externally, using CONVOLV, if he desires.

The subroutines are written in FORTRAN-63, a programming language of the CDC 1604-B computer. Non-standard external symbols appearing in the absorptive routines are ERASE and COOL. Calling ERASE (N,X) zeroes N elements of array X. Calling COOL (LN, X, SIGN) Fast-Fourier transforms the complex array X, to frequency if SIGN = -1.0, to time if SIGN = + 1.0, where LN = \log_2 (number of complex elements in X)(Claerbout et al., 1966). Overall execution time is proportional to K·M [or to (K + KP)·M, if IOPT = 3]. On the 1604, 1.25 sec is required for LAYERS, 12.5 sec for FLAYERS (I), and 18.0 sec for FLAYERS (II), for $K \cdot M = 10^4$. Less time is required if some of the reflection coefficients are zero.

46b

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SUBROUTINE LAYERS( K,C,A, NS,S, M,X, IOPT,KP )
DIMENSION C(K),A(K), S(N), X(M)
REQUIRE DIM. C(K+1), A(K+2). IF IOPT=3, DIM. X(MAX(M,N+KP))
PACKAGE TO SOLVE PLANE LAYER EQU. S. *MAHR WIDTH, REV. AUG. 1970*
IF IOPT = 1, LOWER HALF-SPACE, X = P S / A .
2, FAR SURFACE SOURCE, X = P S / A* .
3, CONTINUED SOURCE, X = P S A1* / A* ,
KP = DEPTH OF SOURCE, TRAVEL-TIME UNITS FROM FAR SURF.
4, BURIED-FCVR OPER., X = S A* / P ,
5, BURIED-SOURCE OPER., X = S A* / P ,
INPUT C = REFLECTION COEFF'S OF K LAYERS, NEAR-FAR, NO SURFACES,
S = SOURCE FUNCTION ON SURFACE SEISMogram (IOPT=4),
OUTPUT X = SEISMogram OF FAR-SURFACE EQUIV. SOURCE (IOPT=5),
A IS AN AUXILIARY ARRAY. MAY EQUIVALENT (C,X) UNLESS IOPT=3,
K1 = N + 1           * C(K1) = 1,
GO TO (10,20,190,210,220), IOPT

10 K1 = N
20 CALL HCTDA( K1,C,A, K )
CALL POLYDIV( N,S, K1,A, M,X )
RETURN

100 KP1 = KP           * KS = K1-KP
S(KS) = 0.           * NP = N + KP
CALL HCTDA( KP1,C(KS+1),A, U )
CALL CONVOLVE( N,S, KP1,A, NP,X )
CALL HCTDA( K1,C,A, KS-1 )
CALL POLYDIV( NP,X, K1,A, M,X )
RETURN

210 S(K1) = -1,
220 CALL HCTDA( K1,C,A, K )
CALL CONVOLVE( N,S, K1,A, M,X )
RETURN
END

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46c

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SUBROUTINE RGTUAC ( K, U, A, KL )
DIMENSION C(K), A(K)
D(M), A(K+1)
DATA LAYER VELTOP *A* FROM REFLECTION COEFFS *B*
I = P = 1.
DO 10 J = 1,K
IF( C(J) ) 10,100
10 IF( J.GT.K ) 30,20
20 D = P / (1,- C(J))
30 JP2 = J + 2      &   JH = JPC / C
DO 50 I = 2,M
JP2M1 = JP2-1
SAV = A(JP2M1) - C(J)*A(I)      &   A(I) = A(I) - C(J)*A(JP2M1)
50 A(JP2M1) = SAV
60 A(I+1) = -C(J)
K = K + 1
DO 200 J = 1,K
70 A(I) = A(J) + P
RETURN
END

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SUBROUTINE CONVOLVE ( M, N, MN, XY )
DIMENSION X(M), Y(N), XY(MN)
DOFS TIME DOMAIN CONVOLUTION,    XY(L) = X(L) * Y(L)
DO 100 J = 1,MN
JP1 = J + 1      &   XY(J) = 0.
100 IF( J.GT.M ) 100,20
200 LI = N
300 IF( J.GT.N ) 400,50
400 IS = JP1-N      &   GU TU 30
500 IS = 1
600 DO 100 I = IS,LI
700 XY(J) = XY(J) + X(I)*Y(IS-1)
800 RETURN

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ENTRY POLYDIV
DOFS POLYNOMIAL DIVISION,    XY(L) = X(L) / Y(L).
Y1 = 1./ Y      &   XY = X * Y1
DO 200 J = 2,MN
JP1 = J + 1
110 XY(J) = Y.      &   GU TU 130
120 XY(J) = X(J)
130 IF( J.GT.N ) 130,160
140 LI = N      &   GU TU 170
150 LI = N
160 LI = 0
170 DO 190 I = 2,LI
180 XY(J) = XY(J) - Y(I)*XY(JP1-I)
190 XY(J) = XY(J) * Y1
200 RETURN
END

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46d

SUBROUTINE FLAYERS(K,C,D, M,X,A,B,F, IOPT,KP)
 DIMENSION C(K),D(K), X(M),A(M),B(M),F(M)
 REQUIRE DIM. C(K+1),D(K+1), A(2M),B(2M). IF IOPT=3, DIM. B(2M)

 PACKAGE TO SOLVE PLANE LAYER EQU.S. *MARK WIRTH OCT 1970*
 REQ-UOM. SOLN W/ LINEAR ABSORPTION LAW. F = E^(-WTD/2), D = 1/0
 K MAY BE ANY NUMBER. M MUST BE A POWER OF 2, M ≥ K
 M CONTAINS DISPERSION, A DUES NOT. (H=U VERSION)
 IF IOPT = 1, LOWER HALF-SPACE. X = F / A .
 2, FAR SURFACE SOURCE. X = F / A .
 3, CONTAINED SOURCE. X = F A1* / A .
 KP = DEPTH OF SOURCE. TRAVEL-TIME UNITS FROM FAR SURF.
 4, BURIED-HCVR OPER., X = A* / F .
 5, BURIED-SOURCE OPER., X = A* / F .

 INPUT C = REFLECTION COEFFS OF K LAYERS, NEAR-FARS, NO SURFACES.
 D = INVERSE U.S OF K+1 LAYERS; U(I) LYING ABOVE C(I).
 OUTPUT X = SEISMOGRAM OR FAR-SURFACE EQUIV. SOURCE (IOPT=5).
 # H, S, F ARE AUXILIARY ARRAYS. MAY EQUIVALENCE (F,X)

 K1 = K + 1 S C(K1) = 1;
 GO TO (10,20,100,210,220), IOPT!

 10 K1 = K S D(K+1) = U.
 20 CALL RCAFTOAL(K1,C,D, M,A,F, K)
 CALL POLYDIV(M,F, M,A, M,X)
 RETURN

 100 KS = K1 - KP S C(KS) = 0.
 CALL RCAFTOAL(K1,C,U, M,A,B, KS-1)
 CALL POLYDIV(M,B, M,A, M,B)
 CALL RCAFTOAL(KP,C(KS+1),0(KS+1), M,A,F, U)
 CALL CONVOLVE(M,B, M,A, M,X)
 RETURN

 210 S(K1) = 1
 220 CALL RCAFTOAL(K1,C,U, M,F,A, K)
 CALL POLYDIV(M,F, M,A, M,X)
 RETURN
 END

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      SUBROUTINE RCAFTOA( K,C,D, M,A,AP, KL )
      DIMENSION C(K),D(K), A(M),AP(M), HA(2)
      TYPE COMPLFX A,AP,Z,DZ,ZFAP, CONJG, DZF
      SUBDIVALNCE (JL,RH), (RM,LM)

      FM = HM = 1./ M           $   LM = 2000000000000000B
      DWT = 6.2831853072 * FM
      NZ = COSF(DWT) + (0.0-1.0)*SINF(DWT)
      DWT = DWT+.5             $   SMD = 0.          $   P = FM
      MH = P/2 + 1              $   L = FM
      DO 1 J = 1,MH
      A(I) = AP(J) = L
      DO 100 I = 1,M
      IF( I.GT.KL ) 20,10
      S = P * (1.-C(I))       $   SMD = SMD + U(I)
      Z = (1.,0.)
      20 100 J = 1,MH
      ZFAP = Z * AP(J)
      IF( C(I) ) 30,25
      AP(J) = ZFAP            $   DO 10 100
      AP(J) = -A(J)*U(I) + ZFAP
      A(I) = A(J) - ZFAP*U(I)
      Z = Z * DZF

      COMPUTE PHASE OF TRANSMISSION FILTER
      SMD = SMD + D(KL+1)     $   L = (0.0-15917494309) * SMD
      CALL CHASE( 2*M,AP )
      DO 150 J = 2,MH,2
      AP(J) = 1. * (1. / (J-1)**2)
      AP(M-J+2) = -AP(J)
      CALL COOL( LM,AP, +1.0 )
      AP = E = P               $   EE = EXP(-DW1*SMD+.5)
      DO 200 J = 2,MH
      S = F * EE
      RA(1) = COSF(AP(J))    $   HA(1) = SINF(AP(J))
      AF(J) = DZ * E           $   AP(M-J+2) = CONJG(AP(J))
      AP(M-J+2) = CONJG( AP(J) )
      CALL COOL( LM,AP, +1.0 )
      CALL COOL( LM,AP, +1.0 )
      MH = MH - 1              $   I = 1
      DO 300 J = 1,MH
      RA(1) = A(I)             $   HA(2) = A(I+1)      $   A(J) = DZ
      RA(1) = AP(I)             $   HA(2) = AP(I+1)      $   AP(J) = DZ
      I = I + 2

      RETURN
      END

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SUBROUTINE FLAYERS(K,C,D, M,X,A,B,F, IOPT,KP)
 DIMENSION C(K),D(K), X(M),A(M),B(M),F(M)
 REQUIRE DIM. C(K+1),D(K+1), A(2M),B(2M). IF IOPt=3, DIM. B(2M)

PACKAGE TO SOLVE PLANE LAYER EQUATIONS. *MARK WIRTH OCT 1970*
 PREQ-DUM. SOLN W/ LINEAR ABSORPTION LAW, $F = E^{(1-WT)/2}$, $D = 1/0$
 K MAY BE ANY NUMBER, M MUST BE A POWER OF 2, M ≥ K
 BOTH F AND A CONTAIN DISPERSION. (LOW-Q VERSION)
 IF IOPt = 1, LOWER HALF-SPACE, $X = F / A$.
 2, FAR SURFACE SOURCE, $X = F / A^*$.
 3, CONTAINED SOURCE, $X = F A_1^* / A^*$,
 KP = DEPTH OF SOURCE, TRAVEL-TIME UNITS FROM FAR SURF.
 4, BURIED-HCVR OPER., $X = A^* / F$.
 5, BURIED-SOURCE OPEN., $X = A^* / F$.
 INPUT C = REFLECTION COEFF. OF K LAYERS, NEAR-FAR, NO SURFACES.
 OUTPUT D = INVERSE Q.S OF K+1 LAYERS, U(I) LYING ABOVE C(I).
 & B, & F ARE UNIQUE AUXILIARY ARRAYS.

K1 = K + 1 C(K1) = 1.
 GO TO (10,20,100,210,220), IOPt

10 K1 = K D(K+1) = U.
 20 CALL HCAFTOAD(K1,C,D, M,A,B,X, K)
 CALL POLYDIV(M,F, M,A, M,X)
 RETURN

100 KS = K1 - KP C(KS) = 0.
 CALL HCAFTOAD(K1,C,D, M,A,B,X, KS+1)
 CALL POLYDIV(M,B, M,A, M,B)
 CALL HCAFTOAD(KP,C(KS+1),U(KS+1), M,A,B,X, U)
 CALL CONVOLVE(M,B, M,A, M,X)
 RETURN

210 D(K1) = -1.
 220 CALL HCAFTOAD(K1,C,D, M,F,A,X, K)
 CALL POLYDIV(M,F, M,A, M,X)
 RETURN
 END

